The Response of Glassy Systems to Random Perturbations: A Bridge Between Equilibrium and Off-Equilibrium

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We discuss the response of aging systems with short-range interactions to a class of random perturbations. Although these systems are out of equilibrium, the limit value of the free energy at long times is equal to the equilibrium free energy. By exploiting this fact, we define a new order parameter function, and we relate it to the ratio between response and fluctuation, which is in principle measurable in an aging experiment. For a class of systems possessing *stochastic stability*, we show that this new order parameter function is intimately related to the static order parameter function, describing the distribution of overlaps between clustering states. The same method is applied to investigate the geometrical organization of pure states. We show that the ultrametric organization in the dynamics implies static ultrametricity, and we relate these properties to static *separability*, i.e., the property that the measure of the overlap between pure states is essentially unique. Our results, especially relevant for spin glasses, pave the way to an experimental determination of the order parameter function.

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I. INTRODUCTION

Ergodicity breaking in glassy systems is well understood within mean-field theory.[1–4] In this context the description of thermal equilibrium has several peculiar aspects. While in non-disordered systems it is usually possible to select pure phases via boundary conditions or suitable external fields, this is problematic in glassy systems, since one does not have in general the a priori knowledge of the low free-energy configurations. Consequently, the Gibbs measure for a large finite system will generically turn out to represent a mixed state. This phenomenon was first discovered and understood in the mean-field theory of spin glasses, and was named "replica symmetry breaking" (RSB) from the technique originally used to analyze these models. However it is not specific to disordered system, and recent theoretical studies have shown that it appears in first-principle computations of the vitrification of simple liquids.[5] Numerical simulations unambiguously show that detailed predictions of the mean-field scenario apply to a number of finite dimensional systems.[6] An experimental verification of the most characteristic predictions of the theory in real materials is however still missing. The main difficulty lies in the fact that it appears impossible to measure directly the basic quantities involved in the theory. One of the most important predictions of the theory is the existence of infinitely many pure states, which depend stochastically on the realization of the quenched disorder[2]—if present—and on system size.[2,7] In this situation one is forced to adopt a probabilistic description of the Gibbs measure, and to study the statistical properties of pure states. A first fundamental characterization is given by the "overlap probability function" $P(q)$,[8] which is the probability distribution function of the codistance among equilibrium states, averaged over the disorder and/or the system size. However, its direct measurement is opposed by two basic difficulties. First, it would require a detailed knowledge of the microscopic configurations of the system. Second, glassy systems are never at equilibrium on experimental time scales. In this paper, we show that linear response theory allows to define a new function $\tilde{P}(q)$ in terms of a set of responses to some appropriate perturbations in the Hamiltonian (i.e., generalized susceptibilities), and to relate them in turn to some dynamical functions which can be experimentally measured. We then argue that, for a large class of systems, $\tilde{P}(q)$ is directly related to the equilibrium order parameter function $P(q)$. These systems, which we shall call "stochastically stable," are such that their "metastate" (i.e., the probability distribution of the pure states) is stable under some random local perturbations of the Hamiltonian.

Linear response theory, i.e., the study of the effects of small perturbations on physical systems, has a long history in statistical physics. One of

its fundamental results is the fluctuation-dissipation theorem (FDT), which establishes, under the hypothesis of thermal equilibrium, the equality of two quantities of a priori very different nature: the response to a small disturbance and the correlation function describing the statistical fluctuation of the disturbed variables. The theorem is of course of fundamental importance: in equilibrium conditions one can get information about the fluctuations of the system just by studying its response to a perturbation. This relation, however, has no reason to hold in general off-equilibrium situations, where the response and the correlation functions are unrelated. In recent works, in fact, several mean-field models of spin glasses and glasses have been studied, in which the asymptotic violation of time-translation invariance appears together with the violation of the equilibrium relation between response and fluctuations. Slow, history-dependent dynamics, usually referred to as *aging*, is a constitutive property of glassy systems.[9] It has been observed in a number of experimental systems, which go from polymeric glasses to charge-density waves, and in the numerical simulations of practically all the glassy systems where it has been searched. We shall mostly refer in this paper to spin glasses as the prototypes of aging systems[10] although our considerations can be applied to any glassy system. We notice in particular that one could discuss in a similar way some systems which are slightly out of equilibrium due to the presence of a slowly time-dependent Hamiltonian or small non-conservative forces,[11, 12] but we shall not develop this discussion here.

In a typical aging experiment one monitors the relaxation of a spin glass at low temperature, after a quench from the high-temperature phase at an initial time. One of the most striking features of aging dynamics is that even for the longest observation times the system does not reach a stationary state. One can concentrate on two classes of observables: "one-time observables," that depend on the configuration of the system at a single time, and "two-time observables," like correlation and response functions, that depend on two times. Due to the relaxational nature of the dynamics, one-time observables tend to time-independent limits. The off-equilibrium nature of the dynamics is manifest in the behavior of some two-time observables, where one sees that time-translation invariance is never achieved, and that there is always a residual dependence on the age of the system.

Our starting point will be the observation that, even in the presence of aging, in short-range systems with non-singular interactions, the long-time limit of the energy density must coincide with its equilibrium average. We then explore the consequences of linear response theory supposing equilibration of the energy density, but not necessarily full thermal equilibrium. Using some special random local perturbations of the original Hamiltonian, we identify a set of susceptibilities which are endowed with very useful properties.

When evaluated at equilibrium, these susceptibilities define the moments of an order parameter function $\tilde{P}(q)$ which is self-averaging. On the other hand, in the dynamical framework, one can show that their dynamic expectation values converge at large times to their static one, establishing the bridge between aging and static ergodicity breaking. We will then establish, for stochastically stable systems, the identity of the *fluctuation dissipation ratio* (FDR), which expresses the asymptotic violation of FDT, with the overlap probability function that describes the statistics of distances among pure states in the equilibrium measure. This result is very general, since it holds in any system with short-range interactions. (In long-range models, or in models with singular interactions, the convergence of the energy density to equilibrium is not guaranteed and the theorem does not necessarily apply.) For equilibrium systems it reduces to the usual relations connecting static and dynamic susceptibilities, which are consequences of the FDT. It is more interesting in the case of glassy systems, where new consequences can be derived.

Similar arguments can be developed in order to study other aspects of ergodicity breaking. Particularly interesting is the geometrical organization of equilibrium states in phase space, for which mean-field theory predicts ultrametricity. This property has a dynamical analog in the ultrametric relation among (two-time) correlation functions for three different long times. Indeed, ultrametricity appears rather naturally in the dynamics within the "weak ergodicity breaking" scenario. We shall show that if ultrametricity holds in dynamics, then stochastic stability implies ultrametricity for the equilibrium system.

A first partial account of our results has appeared in ref. 13.

The paper is organized as follows: In the next section we recall some basic facts about the out of equilibrium dynamics land aging. Section III introduces the overlap probability function and reviews some of the main equilibrium properties. Section IV introduces the set of random local perturbations of the Hamiltonian which are used in order to define the new order parameter function $\tilde{P}(q)$, and its dynamic counterpart. We then develop, in Section V, the notion of stochastic stability, and show the close link which exists between the functions $P(q)$ and $\tilde{P}(q)$ in stochastically stable systems. Section VI establishes the ultrametricity property of these systems, firstly in a dynamical framework, and consequently in the statics. Section VII contains some discussion and summary of the results. Two appendices explain some technical details of the computations.

II. OFF-EQUILIBRIUM DYNAMICS

We use for definiteness the language of magnetic systems, considering classical spins S_x on a d-dimensional lattice of size $N = L^d$. The spins are

real variables in a double-well potential, and we often implicitly take the Ising limit. We do not need to specify much of the interactions, but we assume that they are expressed by a *short-range* Hamiltonian $H(S)$. The boundary conditions will be assumed to be uncorrelated from the energy landscape of the system, for instance they could be periodic.

We suppose for simplicity that the dynamical evolution of the system of N spins is given by the Langevin equation

$$\dot{S}_x = -\frac{\partial H}{\partial S_x} + \eta_x \tag{1}$$

where x is a discrete index that labels the spins and η_x is a white noise of variance $\langle \eta_x(t)\, \eta_y(t') \rangle = 2T\delta_{xy}\delta(t-t')$. We use here the same notation, namely the angular brackets, to denote both the average over the thermal noise in the dynamic context, and the Gibbs average in the static context. Which one is meant should be clear from the context. The use of the Langevin dynamics is not crucial for the statements we shall make below, since they are essentially based on linear response theory. We show in Appendix A how our arguments can be generalized to any system with dissipative dynamics (e.g., Glauber or Metropolis), where the spins are locally updated according to the value of the instantaneous molecular field.

We denote by $\mathbf{S}$ the microscopic configuration of the system: $\mathbf{S} = \{S_x\}$. We make use in the following of the Martin–Siggia–Rose representation of the Langevin dynamics, in which the thermal averages of the observables O are expressed by functional integrals:

$$\langle O(t) \rangle = \int \mathscr{D}(\mathbf{S})\, \mathscr{D}(\hat{\mathbf{S}})\, e^{I[\mathbf{S},\, i\hat{\mathbf{S}}]} O[\mathbf{S(t)}] \tag{2}$$

with the dynamical action

$$I[\mathbf{S}, i\hat{\mathbf{S}}] = \int dt' \sum_x i\hat{S}_x(t') \left[\dot{S}_x + \frac{\partial H}{\partial S_x} + iT\hat{S}_x \right] \tag{3}$$

When inserted in a correlation function, the auxiliary field $i\hat{S}_x(t)$ acts as the functional derivative with respect to a magnetic field at site x and time t.

We suppose that the system has been quenched at time $t = 0$ from a very high temperature into its spin glass phase. It then evolves at a fixed temperature T, starting from a random initial configuration. The fact that the system is off equilibrium appears at long times in the "two time observables," i.e., in the correlation and response functions. We focus on the spin–spin autocorrelation function $C(t, t') = (1/N) \sum_x \langle S_x(t)\, S_x(t') \rangle$, and on

the associated response of the spins at time t to an instantaneous field at time t': $R(t, t') = (1/N) \sum_x \delta\langle S_x(t)\rangle/\delta h_x(t') \equiv (1/N) \sum_x \langle S_x(t)\, i\hat{S}_x(t')\rangle$. In our notations it is understood that the first time argument is always larger then the second one. A quantity which is of particular experimental interest is the relaxation function, which is proportional to the integrated response function,[5]

$$\chi(t, t') = T \int_{t'}^{t} dt'' R(t, t'') \tag{4}$$

In the linear response regime this function measures the adimensional magnetization, deriving from a uniform field acting from time t' to time t. We suppose in the following that the contributions due to the response of a spin to the field in different sites sum up to zero, otherwise our definitions must be adapted to take into account this contribution.

We are interested in the behavior of these quantities in the double long-time limit $t, t' \to \infty$, always taken after the thermodynamic limit $N \to \infty$. The phenomenon of aging is associated with different behaviors, depending on the path along which the two times are sent to infinity.[14] A first regime is obtained when the long-time limit is taken keeping the difference $\tau = t - t'$ finite. In this regime, the correlation function $C(t, t')$ becomes a function of τ: $c(\tau)$. The quantity $q_{EA} = \lim_{\tau \to \infty} c(\tau)$ is known as the Edwards–Anderson order parameter in spin-glass theory. In systems which have only one equilibrium phase and consequently reach equilibrium very fast, the only non-trivial long-time regime corresponds to this limit, and one can measure q_{EA} by taking the long-time limit of $C(t, t')$ in any order, provided $t - t' \to \infty$.

On the other hand, in systems exhibiting aging there are other interesting dynamical regimes, where the correlation relaxes below q_{EA}, and spans the whole interval $[0, q_{EA}]$, depending on how the infinite-time limit is taken.[6] This phenomenon has received the name of "weak ergodicity breaking."[15] It can be generically expected whenever there are different phases in competition. For example, it is found in the domain-growth dynamics of ferromagnets, or more generally in spinodal decomposition. In these cases, if the initial conditions do not favor one phase with respect to the others, one finds for long times $C(t, t') \approx c^*(L(t')/L(t))$, while $\chi(t, t') \approx \chi_\infty + L(t)^{-a} \chi^*(L(t')/L(t))$.[16, 17] This dependence can be understood on the basis of dynamical scaling. After a short transient the system enters an

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[5] The definition of χ adopted here differs by a factor T from the most common one.

[6] Cases in which the correlation does not decay to zero, but to a minimum value $q_0 < q_{EA}$ are also possible. For simplicity, we always consider in this paper $q_0 = 0$. Considering the generic case would just make notations heavier.

asymptotic regime where the growing domains of the different phases are characterized by a typical length $L(t)$. Scale invariance implies for the correlations $C(t, t') \approx C^*(L(t')/L(t))$. The factor $L(t)^{-a}$ (a is equal to 1 in the simplest cases) appearing in χ is due to the fact that only the spins at the boundaries between different domains give a contribution to the long-time response. In pure ferromagnets one has $L(t) \sim t^{1/2}$ for non-conserved order parameter dynamics and $L(t) \sim t^{1/3}$ for conserved dynamics. For domain growth in the presence of some pinning disorder, the growth of $L(t)$ will be slower, but the general behavior should be the same as in pure ferromagnets, with a response function which is much smaller than the correlation at large times.

The case of glassy systems, where aging seems to have a very different nature than in domain-growth processes, is more interesting. It is important, in the following discussion, to consider aging in the linear response function $\chi(t, t')$. Mean-field theory predicts that, asymptotically, C and χ should depend on the same function of t and t', and therefore, that $\chi(t, t') \approx \hat{\chi}(C(t, t'))$. Such a behavior cannot take place in domain growth kinetics (nor in a droplet description of aging), in which only the spins at the border of the growing domains contribute to the response. In order to characterize the relation among correlation and response one can define the function

$$X(q) = -\frac{\partial \hat{\chi}}{\partial q} = - \lim_{\substack{t, t' \to \infty \\ C(t, t') = q}} \frac{\partial \chi(t, t')}{\partial t'} \bigg/ \frac{\partial C(t, t')}{\partial t'}$$

$$\equiv \lim_{\substack{t, t' \to \infty \\ C(t, t') = q}} TR(t, t') \bigg/ \frac{\partial C(t, t')}{\partial t'} \tag{5}$$

which is called the fluctuation-dissipation ratio, and depends a priori on the value q of the correlation which is kept fixed while one sends the two times t, t' to infinity.

In the finite τ regime, in which $q \in [q_{EA}, 1]$, local equilibrium implies $X(q) = 1$, which is the expression of the usual fluctuation-dissipation theorem. But a non-trivial behavior is possible for $q \in [0, q_{EA}]$. Mean-field theory predicts that the function dX/dq is non-trivial, and satisfies the properties of a probability distribution, a fact that had remained unexplained up to now. Simulations have shown that this fact also holds in 3D and 4D spin glasses.[16] The function $X(q)$ is experimentally accessible through waiting-time monitored measurements of the thermoremanent magnetization and of the noise. Its measurement will allow one to distinguish whether the glassy behavior is a slow domain growth,

where $X(q) = \theta(q - q_{EA})$, or whether it has a more complicated behavior, with $X(q) \neq 0$ at $q < q_{EA}$, as seen in mean field and in simulations.

III. EQUILIBRIUM

For short-range Hamiltonians with non singular interactions, classical arguments, based on neglecting surface with respect to volume terms, show that the averages of the free energy density have a thermodynamic limit. Intensive quantities, which can be obtained as derivatives office free energy densities are unique and self averaging except, in case, at first order phase transitions points, where more than one phase is present. We are interested in situations of broken ergodicity, where the infinite-volume equilibrium measure is not unique. Sometimes, when one knows a priori the set of equilibrium states, one can select one element of the set of the equilibrium measures or another by properly choosing the boundary conditions or some small external field. In ordered systems this allows to select *pure phases*, or ergodic components, which correspond to physical equilibrium states. Unfortunately, this procedure is of no use in the case of glassy systems, where our lack of knowledge on the structure of the low lying states prevents us from picking up the correct boundary conditions necessary to project on the pure phases. Indeed, pure phases have identical values of the densities of extensive quantities like internal energy, magnetization etc.

If the boundary conditions are uncorrelated with the energy landscape (e.g., periodic) the Gibbs measure turns out to be a mixed measure. In addition the measures can have a strong dependence on the quenched disorder and/or on the system size. Under these conditions it is useful to study the statistical properties of the mixtures and to perform the analysis at very large, but finite N, avoiding the discussion of pure states for an actually infinite system. An order parameter function which shows up naturally in mean field is the overlap probability function. It is defined by picking up two microscopic configurations $\mathbf{S}$ and $\mathbf{S}'$ at random, with the Gibbs measure, and measuring the histogram $P(q)$ of their overlap $Q(\mathbf{S}, \mathbf{S}') = (1/N) \sum_x S_x S_x'$. More precisely the order parameter function, $P(q)$, is given by

$$P(q) = \lim_{N \to \infty} E_J \left[\frac{1}{Z^2} \sum_{\mathbf{S}, \mathbf{S}'} \exp(-\beta(H(\mathbf{S}) + H(\mathbf{S}'))) \, \delta(Q(\mathbf{S}, \mathbf{S}') - q) \right] \qquad (6)$$

We have denoted by E_J the average over the quenched disorder in the system (if present) or over a window in N, increasing with N, and by Z the

partition function. In ergodic systems, $P(q)$ is given by a single delta function, whereas it can have a nontrivial structure in systems with ergodicity breaking. This appears explicitly in several mean-field models. Present numerical simulations give evidence in favor of ergodicity breaking and of a non-trivial $P(q)$ in finite-dimensional spin-glass models,[6] but the relevance of this result has been challenged.[18] The theorem we are going to discuss relates the function $P(q)$, which is not directly accessible in experiments, to the fluctuation dissipation ratio $X(q)$, which is measurable. It thus clarifies the relevance of the function $P(q)$ defined by Eq. (6) and opens the way to obtain some experimental evidence concerning its behavior.

In the next section we shall need to compute some correlation functions like

$$\mathscr{C}_p = E_J[\langle S_{x_1} \cdots S_{x_p} \rangle^2]$$
$$= E_J\left[\frac{1}{Z^2} \sum_{\mathbf{S}, \mathbf{S}'} \exp(-\beta(H(\mathbf{S}) + H(\mathbf{S}'))) \, S_{x_1} S'_{x_1} \cdots S_{x_p} S'_{x_p} \right] \qquad (7)$$

for very far apart sites $x_1,..., x_p$, in the presence of some quenched random variables J. The relation of these moments with physically clustering states and its expression in terms of the function $P(q)$ is discussed in ref. 2. We just give here the result:

$$\mathscr{C}_p = \int dq \, P(q) \, q^p \qquad (8)$$

IV. PERTURBING THE HAMILTONIAN: A NEW ORDER PARAMETER FUNCTION

In this section we shall define a new equilibrium order parameter function $\tilde{P}(q)$, and the corresponding fluctuation-dissipation ratio $\tilde{X}(q)$. The derivation of their properties involves the study of the linear response to some special sets of perturbations of the original Hamiltonian. This method has been recently used to derive interesting properties of the overlap distribution at equilibrium.[19, 20] We start by recalling some well-known perturbations which are long ranged, and whose expectation values yield the moments of the function $P(q)$ in statics and of dX/dq in dynamics. However, the long-range nature of the perturbation forbids one to establish the relationship between the long time dynamics and the statics. We shall thus introduce a new set of perturbations which are purely local, and from which the result can be derived.

It is known that the moments of $P(q)$ and dX/dq are respectively related to the canonical[19] and the dynamical averages[21] of long-range perturbations to the Hamiltonian of the form

$$H_p^{\text{LR}}(S) = \sum_{i_1 < \cdots < i_p}^{1, N} J_{i_1, \ldots, i_p} S_{i_1} \cdots S_{i_p} \tag{9}$$

where the couplings $J_{i_1, \ldots, i_p}$ are independent Gaussian variables with zero mean and variance $E_J(J_{i_1, \ldots, i_p}^2) = p!/(2N^{p-1})$. Now, one can easily see that the canonical average of H_p^{LR} with a perturbed Hamiltonian given by

$$H_\varepsilon = H_0 + \varepsilon H_p^{\text{LR}} \tag{10}$$

verifies, for all values of ε,

$$E_J \langle H_p^{\text{LR}} \rangle = -\beta \varepsilon N \left(1 - \int dq \, P_\varepsilon(q) \, q^p \right) \tag{11}$$

irrespective of the specific form of H_0. Here the function $P_\varepsilon(q)$ is the order parameter function (6), in the presence of the perturbing term in the Hamiltonian. The derivation, reported in Appendix A, involves an integration by parts in a finite system, followed by the infinite-volume limit.

On the other hand, in off-equilibrium dynamics, one has

$$\lim_{t \to \infty} E_J \langle H_p^{\text{LR}}(t) \rangle = -N \beta \varepsilon \left(1 - \int dq \, \frac{dX_\varepsilon(q)}{dq} \, q^p \right) \tag{12}$$

where $X_\varepsilon(q)$ is the FDR in the presence of the perturbation (we recall that the system has been quenched at time zero). For completeness, formulae (11) and (12) are derived in Appendix A.

These identities suggest that one could have $P(q) = dX/dq$. However, in order to derive this result one needs two more steps which are far from trivial. First, one should prove that H_p^{LR} is statically self-averaging and that its dynamic expectation value tends to the equilibrium one. Second, one must discuss the continuity of the functions at $\varepsilon = 0$. Concerning the first point, our main concern is that the long-range perturbation could give rise to infinitely long-lived metastable states in the system, so that $\lim_{\to \infty} \langle H_p^{\text{LR}}(t) \rangle_{\text{dyn}} \neq \langle H_p^{\text{LR}} \rangle_{\text{stat}}$. Although we feel that this is not likely to happen, in most systems, for small enough ε, we do not know at present how to prove this fact. Below we use local (short-range) perturbations which Bet around this problem, and in the next section we address the issue of continuity at small ε.

The strategy we follow amounts to modify the definition of the perturbations, introducing a new H_p which is an extensive sum of some local observables. Then the whole Hamiltonian $H_0 + \varepsilon H_p$ is short range, and thus self-averaging, and the long-time equilibration of the dynamical expectation $\langle H_p(t) \rangle$ towards its equilibrium value can be shown as follows. One starts by proving that the free energy equilibrates. The classical proof of this fact is based on nucleation arguments. One first notices that the time-dependent free-energy density must reach, for long times, the equilibrium value f_{eq}. All metastable states with a higher free-energy density f^* can be destabilized by the nucleation of a bubble of radius r with a free energy cost equal or lower than $cr^{d-1} + (f_{eq} - f^*) r^d$. Therefore the free energy density reaches its equilibrium value. Since the static perturbation energy $\langle H_p \rangle$ is the derivative of βF with respect to ε, the equilibration of the free energy also implies the equilibration of $\langle H_p(t) \rangle$.[7] This differentiation will be allowed at small ε provided the free energy is regular enough at small ε, which is the issue which we shall discuss in the next section.

The local perturbing Hamiltonian can be chosen in several ways, we present here the one we already considered in ref. 13. Let us consider a set of operators which translate the lattice along a direction $\hat{\imath}$ parallel to one of the coordinate axes:

$$\mathcal{T}_k^{(p)}(x) = x + \frac{kL}{p} \hat{\imath}; \qquad k = 1,..., p-1 \tag{13}$$

We denote by $\mathcal{S}_1$ the set in which the coordinate in the direction $\hat{\imath}$ takes the values $1, 2,..., L/p$. The new perturbations will have the form

$$H_p(\mathbf{S}) = \sum_{x \in \mathcal{S}_1} J_x^{(p)} S_x S_{\mathcal{T}_1^{(p)}(x)} \cdots S_{\mathcal{T}_{(p-1)}^{(p)}(x)} \tag{14}$$

where the couplings $J_x^{(p)}$ are independent, identically distributed Gaussian variables, with zero mean and variance $E_J J_x^2 = p$. At first sight, also these perturbations seem to contain long-range interactions. However, they can be viewed as short-range observables in a different space. This is more easily illustrated for $p = 2$, but the generalization to arbitrary p is immediate. For $p = 2$ we can divide the space into two halves ($\mathcal{S}_l$ and $\mathcal{S}_r$) and rename the spins in the right-hand half so that if $x \in \mathcal{S}_l$ then $\mathcal{T}(x) \in \mathcal{S}_r$

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[7] Generically, one should qualify this statement at first-order phase transition points, where the perturbation can select a given phase. However, as previously noticed, we are interested in a situation in which all pure phases have the same values of extensive observables, and this does not happen.

and $S_{\mathcal{T}(x)} = S'_x$ (we wrote $\mathcal{T}(x)$ for $\mathcal{T}_1^{(2)}(x)$). The total Hamiltonian can now be written as

$$H(\mathbf{S}, \mathbf{S}') = H_l(\mathbf{S}) + H_r(\mathbf{S}') + B(\mathbf{S}, \mathbf{S}') + \varepsilon \sum_{x \in \mathscr{S}_l} J_x^{(2)} S_x S'_x \qquad (15)$$

The Hamiltonians H_l and H_r refer respectively to the spins in $\mathscr{S}_l$ and $\mathscr{S}_r$. The term $B(\mathbf{S}, \mathbf{S}')$ is a surface term whose presence does not affect the average of H_2. Dropping it, the Hamiltonian (15) characterizes a spin system of size $L^d/2$, with two spins S_x, S'_x on each site, and a purely local interaction. The static expectation value of the perturbation $\langle H_2 \rangle$, gives therefore a contribution to the internal energy of the system which is extensive and self-averaging, i.e., independent (in the thermodynamical limit) of the particular realization of the disorder contained in either H_0 or H_2. Its long-time limit in off-equilibrium dynamics must yield its canonical value.

We now take advantage of the self-averaging property of H_2 to express its static and dynamic values as functions of $P(q)$ and $X(q)$ respectively. Let us first discuss the statics:

$$\begin{aligned} \langle H_2 \rangle &= E_J \langle H_2 \rangle \\ &= E_J \left[\int \mathscr{D}(\mathbf{S}) \exp(-\beta(H(\mathbf{S}) + \varepsilon H_2(\mathbf{S}))) \sum_x J_x^{(2)} S_x S_{\mathcal{T}(x)} \right] \end{aligned} \qquad (16)$$

Integrating by parts over the random couplings $J_x^{(2)}$ we find

$$\langle H_2 \rangle = \beta \varepsilon N (1 - E_J \langle S_x S_{\mathcal{T}(x)} \rangle^2) \qquad (17)$$

In the linear response regime we can write

$$E_J \langle S_x S_{\mathcal{T}(x)} \rangle^2 = E_J \langle S_x S_y \rangle^2 + O(\varepsilon) \qquad (18)$$

where x and y are far away sites not directly coupled in the Hamiltonian, and consequently, according to the discussion of the previous section,

$$\langle H_2 \rangle = \beta \varepsilon N \left(1 - \int dq \, P_\varepsilon(q) \, q^2 \right) \qquad (19)$$

where the function $P_\varepsilon(q)$ is the average overlap probability function of the perturbed system. Let us remark that in order to derive (19) we need to integrate over the couplings $J_x^{(2)}$. In the case where also H_0 contains quenched disorder we can choose to integrate over it or not, but the result should be independent of this operation. This tells us that the order

parameter function, averaged over the couplings in H_2, is self-averaging with respect to the quenched variables in H_0, even for infinitesimal ε.

Similar considerations hold in the case of the dynamics. The self-averaging property of H_2 in the dynamics allows us to write

$$\langle H_2(t)\rangle = E_J \langle H_2(t)\rangle$$
$$= E_J \int \mathscr{D}(\mathbf{S}) \, \mathscr{D}(\hat{\mathbf{S}}) \, e^{I[\mathbf{S}, \, i\hat{\mathbf{S}}]} \sum_x J_x^{(2)} S_x(t) \, \mathscr{S}_{\mathscr{T}(x)}(t) \quad (20)$$

Integrating by parts over the $J_x^{(2)}$'s, and observing that the insertion of $i\hat{S}_x(t')$ acts as the derivative with respect to an impulsive magnetic field at site x and at time t' $(i\hat{S}_x(t') \mapsto \delta/\delta h_x(t'))$ we obtain

$$E_J \langle H_2(t)\rangle = 2\varepsilon \sum_x E_J \left[\int_0^t dt' \, \frac{\delta}{\delta h_{\mathscr{T}(x)}(t')} \langle S_x(t) \, S_x(t') \, S_{\mathscr{T}(x)}(t)\rangle \right] \quad (21)$$

In the linear response regime, $\beta\varepsilon \ll 1$, the average of the product on faraway sites factorizes up to terms of order ε, and one has

$$E_J \left[\frac{\delta}{\delta h_{\mathscr{T}(x)}(t')} \langle S_x(t) \, S_x(t') \, S_{\mathscr{T}(x)}(t)\rangle \right] = C(t, t') \, R(t, t') + O(\varepsilon) \quad (22)$$

Assuming that the bound holds uniformly in time and substituting the definition (5) of the FDR, we obtain, for large values of t,

$$2\varepsilon\beta N \int_0^1 dq \, X_\varepsilon(q) \, q = \varepsilon\beta N \left(1 - \int_0^1 dq \, \frac{dX_\varepsilon}{dq} \, q^2 \right) \quad (23)$$

where we have used $C(t, t) = 1$, and $\lim_{t \to \infty} C(t, 0) = 0$ (see footnote 2). We have denoted by X_ε the FDR of the system with the perturbed Hamiltonian H_ε. Comparing the two results, (23) for the dynamics and (19) for the statics, we see that the second moments of the dynamical order parameter function $dX_\varepsilon(q)/dq$ and of the static one $P_\varepsilon(q)$ coincide for the system in the presence of the perturbation εH_2. It is straightforward to generalize this derivation to arbitrary p (for $p = 1$ the perturbation is nothing but a small random field term), which shows that the pth moments of the two functions $dX_\varepsilon(q)/dq$ and $P_\varepsilon(q)$ coincide for small ε. Therefore the two functions coincide, and their limits for $\varepsilon \to 0$, $d\tilde{X}(q)/dq$ and $\tilde{P}(q)$ respectively, also coincide. We have thus found one "new" order parameter function which describes both. the static equilibrium situation and the out of equilibrium aging dynamics. In the next section we will see to what extent one can relate these functions to the FDR and order parameter function of the unperturbed system.

Finally let us point out that the choice of the perturbation is by no means unique. In principle we could also couple different systems instead of translated copies of the same systems. This would amount to introducing p different systems and using as as perturbation

$$H_p(\mathbf{S}) = \sum_x J_x^{(p)} S_x^1 S_x^2 \cdots S_x^p \tag{24}$$

where the unperturbed Hamiltonian is given by the sum of Hamiltonians of the p uncoupled system. This approach is also viable and yields the same results as the one discussed above. Another possibility is to consider just one system with perturbing interactions with a finite range (Kac potential type), to be sent to infinity after the thermodynamic limit.

Let us conclude by noticing *en passant* that we can use the short-range perturbations (14) to repeat the derivation by Ghirlanda and Guerra[22] of the identities for the probability function of two overlaps at equilibrium, avoiding also in this case problems related with the proof of self-averaging and continuity of the correlation functions for long-range perturbations.

V. STOCHASTIC STABILITY

We have seen in the last section that the static and dynamic order parameter distribution functions, respectively $P_\varepsilon(q)$ and dX_ε/dq, are equal in the presence of the perturbation, and that therefore their $\varepsilon \to 0$ limits, $\tilde{P}(q)$ and $d\tilde{X}/dq$, coincide. We now address the question of the relation between these quantities and the conventional order parameter functions $P(q)$ and dX/dq, i.e., the order parameter function and FDR defined in the absence of the perturbation. If we were to apply naive perturbation theory, we would directly conclude that they are equal. However, while this conclusion is inescapable for ergodic systems, it needs to be qualified in the more interesting case in which many ergodic components are present.

Let us start with the statics. The problem is to understand what is the "equilibrium" state in the presence of the perturbation εH_p, and how it is related to the equilibrium state at $\varepsilon = 0$. Some problems can arise whenever the equilibrium expectation value of H_p is not the same for all pure phases of the unperturbed systems. Then the limit value of H_p as $\varepsilon \to 0$ will be the one corresponding to the favored phases. A simple example is the Ising model in the ferromagnetic phase-with free boundary conditions, to which one adds a negative magnetic field term as a perturbation. Then in the vanishing field limit, the long-time limit of the perturbation is evaluated in the state with negative spontaneous magnetization, while the unperturbed

measure corresponds to a mixture of the positive and negative magnetization pure states. Clearly in this case $\tilde{P}(q) \neq P(q)$. These two functions will also differ if the perturbation, instead of being a uniform field, is a random field.

Indeed, the stochastic perturbations that we have considered will in general reshuffle the weights of the different ergodic components in the Gibbs measure, or even change their nature, and this changes the $P(q)$ function to a different one ($\tilde{P}(q)$). However our perturbations are random perturbations which are not correlated with the original Hamiltonian. So they should change the free energies of the various pure states of the original systems by random amounts. Usually in glassy systems the *distribution* of these free energies is stable under independent random increments, as has been shown in mean field (in fact this property lies at the heart of the cavity method[2]). If this is the case then the two functions $\tilde{P}(q)$ and $P(q)$ will coincide. A noticeable exception to this result is the case where the original Hamiltonian has an exact symmetry, which is lifted by the perturbation. The simplest case is that of a spin glass with a Hamiltonian invariant under spin inversion. In this case $P(q) = P(-q)$, since each pure state appears with the same weight as its opposite in the unperturbed Gibbs measure. On the other hand, if we consider H_p with odd p, this symmetry is lifted. This means that in the $\varepsilon \to 0$ limit only half of the states are kept. If the reshuffling of their free energies is indeed random, then we shall have $\tilde{P}(q) = 2\theta(q) P(q) \equiv \hat{P}(q)$. The same type of reasoning applies whenever the overlap q transforms according to a representation of the symmetry group of the unperturbed Hamiltonian H_0.

Let us now discuss the case of dynamics. It is clear that the finite-time response and correlation functions involved in the definition of the FDR are continuous functions of ε for $\varepsilon \to 0$. If this limit is uniform in time, so that the infinite time and the $\varepsilon \to 0$ limits commute, one has $\tilde{X}(q) = X(q)$. On the other hand, it may be the case that the linear response regime shrinks to zero as the time goes to infinity. This possibility shows up in the situation, in which the perturbations favor one (or more phases) phase. However we stick here to random perturbations such that the expectations of H_p vanishes at $\varepsilon = 0$. In this case it is reasonable to assume that the linear response regime survives at very long times, implying that $\tilde{X}(q) = X(q)$. Let us stress that the existence of a linear response regime uniform in time is implicitly assumed in experiments attempting to measure $X(q)$ in real systems, and is anyway a question susceptible of experimental investigation.

Once the effect of exact symmetries is taken into account, one may expect that, for a large class of systems, the limit function $\tilde{P}(q)$ in the limit of small perturbations tends to the order parameter function $\hat{P}(q)$ of the

pure system where the exact symmetries are lifted, what is nothing but a statement of continuity of the correlations at small ε. We call this continuity property of the correlations *stochastic stability*. Ordinary systems with symmetry breaking and mean-field spin glasses are examples of stochastically stable systems. In symmetry breaking systems (and in ergodic systems), the equality of $\tilde{P}$ and $\hat{P}$ is immediate, since both functions consist in a single delta function. Thus, the problem of deriving the equality between $\tilde{P}$ and $\hat{P}$, appears only when the coexisting phases are unrelated by symmetry.

Unfortunately, we are not able to characterize the class of stochastically stable systems in general. In particular, we do not know for sure whether short-range spin glass, for which our theorem is most interesting, belong to this class. However, stochastic stability has been established rigorously in mean field problems,[19, 20] and numerically exhibited in three and four dimensional spin glasses, where also the equality between $P(q)$ and dX/dq has been confirmed.[23]

Stochastic stability is a very powerful property, and it is the ingredient which allows to relate the properties of the low lying configurations which dominate the Gibbs measure, to those of the configurations much higher in energy which are seen in the dynamics. This is most easily explained in the usual framework of replica-symmetry breaking, considering an approximation with only two possible values of the overlap, $q_0 = 0$ among different states and q_1 among the same state (i.e., one-step replica-symmetry breaking). The probability of finding a state with total free energy $F_\alpha = F$ is given by $\rho(F - F_0)$, where F_0 is the equilibrium free energy. The weight of each state is given by

$$w_\alpha \propto \exp(-\beta F_\alpha) \tag{25}$$

In one-step replica-symmetry breaking, the states which contribute to the Gibbs measure have nearly degenerate free energies. The non-extensive fluctuation of their free energies, corresponding to the low F regime of $\rho(F)$, is given by refs. 24 and 25

$$\rho(F) \propto \exp(\beta m F) \tag{26}$$

and the function $P(q)$ is given by

$$P(q) = m\delta(q - q_0) + (1 - m)\,\delta(q - q_1) \tag{27}$$

In its dynamical evolution from a random initial state, the total energy of a configuration at time t decreases as

$$E = E_\infty + cNt^{-\lambda} \tag{28}$$

where λ is an appropriate exponent. Correspondingly the difference between the total free energy at time t and the equilibrium value will be always of order N, with a prefactor going to zero when t goes to infinity. One can physically argue that the dynamics only probes the behavior of the function $\rho(F)$ at large argument, and should not be related to the statics. Stochastic stability solves this apparent paradox, because it forces the function $\rho(F)$ to be of the form (26), not only when $F - F_0$ is finite, but also in the range where $F - F_0$ is *extensive* but small (say of order εN). Indeed it imposes that the form of the function $\rho(F)$ remains unchanged (apart from a possible shift in F_0) when one adds a small random perturbation. Consider the effect of a perturbation of strength ε on the free energy of a state, say α. The unperturbed value of the free energy is denoted by F_α. The new value of the free energy G_α is given by $G_\alpha = F_\alpha + \varepsilon r_\alpha$ where r_α are identically distributed uncorrelated random numbers. Stochastic stability implies that the distribution $\rho(G)$ is the same as $\rho(F)$. Expanding to second order in ε we see that this implies $d\rho/dF \propto d^2\rho/dF^2$, whose only physical solution (apart the trivial one $\rho(F) = 0$, which corresponds to non-glassy systems) is given by Eq. (26). The same conclusion could be obtained using the methods of ref. 26 computing the sample-to-sample fluctuations of the function $P_J(q)$, which in this case, where ultrametricity is trivially satisfied, are completely determined by the knowledge of of the function $P(q)$. We see that stochastic stability fixes the form of the function ρ and therefore connects,in an inextricable way the low and the high free energy part of the function ρ, avoiding a possible paradox.

VI. ULTRAMETRICITY AND SEPARABILITY

A very interesting aspect of mean-field spin glasses at equilibrium is the hierarchical organization of the pure phases, which build an ultrametric space. Indeed, in these models, the probability distribution of three overlaps

$$P^{(3)}(q_{12}, q_{13}, q_{23}) = E_J \frac{1}{Z^3} \sum_{\mathbf{S}, \mathbf{S}', \mathbf{S}''} \exp(-\beta(H(\mathbf{S}) + H(\mathbf{S}') + H(\mathbf{S}'')))$$
$$\times \delta(Q(\mathbf{S}, \mathbf{S}') - q_{12})\, \delta(Q(\mathbf{S}, \mathbf{S}'') - q_{13})\, \delta(Q(\mathbf{S}', \mathbf{S}'') - q_{23}) \tag{29}$$

has been shown to vanish unless its arguments verify the *ultrametric inequality*

$$q_{12} \geqslant \min\{q_{13}, q_{23}\} \tag{30}$$

It would be extremely interesting to measure the function $P^{(3)}(q_{12}, q_{13}, q_{23})$ in real materials. However, it is clear that a direct measure is impossible for the same fundamental reasons as the direct measure of $P(q)$ is impossible: impossibility to reach equilibrium, impossibility to access the microscopic structure of the pure phases.

In the following we first review the dynamical analog of the ultrametric identity. We then provide some arguments in favor of a link between the dynamic ultrametricity and the ultrametricity in the static organization of pure phases, as well as some arguments which indicate that dynamical ultrametricity could be a generic property of stochastically stable systems. The arguments are of the same nature as the ones used in the previous section, in that they rely on the study of systems weakly perturbed by some appropriately chosen random couplings. We need to introduce additional assumptions, which make the conclusion more conjectural.

In glassy dynamics, weak ergodicity breaking scenario implies more than just the decay of the autocorrelation at long times. In particular, it implies a relation among the three different two-time correlation functions that can be built from the configurations of the systems at three different (long) times $t_{\min}$, t_{int}, $t_{\max}$, namely $C(t_{\text{int}}, t_{\min})$, $C(t_{\max}, t_{\min})$, $C(t_{\max}, t_{\text{int}})$. Exploiting the monotonicity of C in both time arguments one can, for each t_{int}, invert the relation between $C(t_{\text{int}}, t_{\min})$ and $t_{\min}$ and the one between $C(t_{\max}, t_{\text{int}})$ and $t_{\max}$. In this way, for each t_{int}, $C(t_{\max}, t_{\min})$ can be expressed as a function of $C(t_{\text{int}}, t_{\min})$ and $C(t_{\max}, t_{\text{int}})$, yielding[27]

$$C(t_{\max}, t_{\min}) = f_{t_{\text{int}}}(C(t_{\text{int}}, t_{\min}), C(t_{\max}, t_{\text{int}})) \tag{31}$$

In the long-time limits keeping fixed $q_{12} = C(t_{\text{int}}, t_{\min})$ and $q_{23} = C(t_{\max}, t_{\text{int}})$, the function $f_{t_{\text{int}}}(q_{12}, q_{23})$ tends to a limit function $f(q_{12}, q_{23})$. The properties of this function have been studied in great detail in ref. 27. The "fixed points" of f, defined as the values of q for which $q = f(q, q)$ are especially relevant. Indeed, if q and q' are fixed points, then f verifies the ultrametric relation $f(q, q') = \min\{q, q'\}$. The scaling form $C(t, t') = c^*(h(t')/h(t))$, where $h(t)$ is some monotonically increasing function of t, holds for values of the correlation between contiguous fixed points. This shows that in the limit where the two times t, t' become large, the corresponding two-time plane can be divided into sectors, called $\mathscr{D}_u$, which we label by the index u. Each sector $\mathscr{D}_u$ is characterized by a monotonically increasing function of t, $h_u(t)$, and is defined by sending the two times t, t' to infinity with a fixed ratio $\lambda = h_u(t')/h_u(t)$. One then has $C(t, t') \to c_u^*(\lambda)$. The various sectors have a hierarchical organization[28] in the sense that, if $t, t' \in \mathscr{D}_u$ and $t', t'' \in \mathscr{D}_v$, then $t, t'' \in \mathscr{D}_{\min(u, v)}$.

So far this structure is just one way of expressing the long time limit of the function of two variables $C(t, t')$, in terms of a (possibly continuous) set of functions of one variable $c_u^*(\lambda)$. It is a very general structure which relies only on weak ergodicity breaking. However this structure becomes much more interesting in view of the explicit solution of specific mean field models, which has shown that the fluctuation-dissipation ratio $X(q)$ is constant in each domain $\mathscr{D}_u$, and correspondingly,

$$\lim_{\substack{t, t' \to \infty \\ h_u(t)/h_u(t') = \lambda}} TR(t, t') \bigg/ \frac{\partial C(t, t')}{\partial t'} = X_u \qquad (32)$$

independently of the value of λ. This really implies a reduction of the number of variables, since it means that the integrated response is just proportional to the correlation in each domain. We will see that this hypothesis has far reaching consequences. We shall define the property of dynamical ultrametricity as the fact of having both the hierarchical domain organization, together with a constant fluctuation dissipation ratio within each domain.

We shall now assume that we have a stochastically stable systems. Using our general strategy of linear response to random perturbations, we first argue that if it is dynamically ultrametric, then it is very likely to be ultrametric in the sense of the equilibrium distribution. We then relate dynamical ultrametricity to the property of *separability*, introduced in ref. 26, which states that pairs of states with a given overlap q cannot be distinguished by the value of any differently defined generalized overlap. For example we can define an "energy overlap" of two configurations as $Q^H(\mathbf{S}, \mathbf{S}') = (1/N) \sum_x h_x(\mathbf{S}) h_x(\mathbf{S}')$, where $h_x(\cdot)$ denotes the molecular field on site x corresponding to a given spin configuration. Separability means that, for any pair (α, β) of pure phases, whose overlap $q_{\alpha\beta}$ is equal to q, q^H is a self-averaging quantity $q^H = f(q)$ that depends only on q. A trivial example of non-separable system, which is neither stochastically stable, is given by the union of two separable systems. Indeed it is clear that

$$q_{\text{tot}}^H = q_1^H + q_2^H = f(q_1) + f(q_2) \neq f(q_1 + q_2) \qquad (33)$$

as long as f is non-linear. Separability is therefore a strong requirement in the sense that it implies strong correlations among different parts of a system.

We now generalize the static-dynamic equalities derived above to the three-point probability distribution function $P^{(3)}(q_{12}, q_{13}, q_{23})$, relevant for the discussion of the ultrametric organization of pure phases. As in the previous case, we must identify a set of susceptibilities generated by this

functional. Using the same construction as in Section IV, let us divide the
lattice in $p = l + m + n$ slices and consider two perturbations of the kind

$$H_p^{(1)}(\mathbf{S}) = \sum_{x \in \mathscr{S}_1} J_x S_x S_{\mathscr{T}_1^{(p)}(x)} S_{\mathscr{T}_2^{(p)}(x)} \cdots S_{\mathscr{T}_{l+m-1}^{(p)}(x)} \qquad (34)$$

and

$$H_p^{(2)}(\mathbf{S}) = \sum_{x \in \mathscr{S}_1} J_x' S_{\mathscr{T}_l^{(p)}(x)} S_{\mathscr{T}_{l+1}^{(p)}(x)} \cdots S_{\mathscr{T}_{p-1}^{(p)}(x)} \qquad (35)$$

with *two* different realizations J_x and J_x' of the quenched couplings. Notice
that $H_p^{(1)}$ couples spins belonging to the first $l + m$ slices, while $H_p^{(2)}$ couples
spins belonging to the last $m + n$ slices. The total Hamiltonian is now
$H_{\varepsilon\varepsilon'} = H_0 + \varepsilon H_p^{(1)} + \varepsilon' H_p^{(2)}$. We now evaluate the average of the observable

$$O(\mathbf{S}) = \sum_{x \in \mathscr{S}_1} J_x J_x' S_x S_{\mathscr{T}_1^{(p)}(x)} \cdots S_{\mathscr{T}_{l-1}^{(p)}(x)} \times S_{\mathscr{T}_{l+m}^{(p)}(x)} \cdots S_{\mathscr{T}_{p-1}^{(p)}(x)} \qquad (36)$$

where the spins in the first l slices are coupled with the spins of the last n
slices. As in the case of the observables H_p, useful expressions for the
dynamic and the static averages of O are obtained by exploiting the self-
averaging property and by integrating by parts over the variables J_x
and J_x'. The calculations are lengthy, but conceptually similar to the ones
relative to the observables H_p. They are reported in Appendix B. We thus
obtain the following expression of the static average:

$$\langle O \rangle = \beta^2 \varepsilon \varepsilon' \frac{N}{p} \left(1 - \int_0^1 dq \, P_\varepsilon(q)(q^{l+m} + q^{l+n} + q^{n+m}) \right.$$
$$\left. + 2 \int_0^1 dq_{12} \, dq_{13} \, dq_{23} \, P_\varepsilon^{(3)}(q_{12}, q_{13}, q_{23}) \, q_{12}^n q_{13}^m q_{23}^l \right) \qquad (37)$$

Correspondingly, in off-equilibrium dynamics, we find the following
expression of $\langle O(t) \rangle$ in terms of the time-dependent correlation and
response functions:

$$E_J E_{J'} \langle O(t) \rangle = \beta^2 \varepsilon \varepsilon' \frac{N}{p} \int_0^t dv \int_0^t du$$
$$\times ([\, lC(t, v)^{l-1} R(t, v)][C(v, u)^m][nC(t, u)^{n-1} R(t, u)]$$
$$+ [\, lC(t, v)^{l-1} R(t, v)][mC(v, u)^{m-1} R(v, u)][C(t, u)^n]$$
$$+ [C(t, v)^l][mC(u, v)^{m-1} R(u, v)][nC(t, u)^{n-1} R(t, u)]) \qquad (38)$$

We now exploit the ultrametric relation among fixed points of the application $f(q, q')$ and the constancy of $X(q)$ for values of q between contiguous fixed points that we discussed in Section II. We can thus express $\lim_{t \to \infty} \langle O(t) \rangle$ as a functional of $X_\varepsilon(q)$, namely

$$
\lim_{t \to \infty} E_J E_{J'} \langle O(t) \rangle
$$

$$
= \beta^2 \varepsilon \varepsilon' \frac{N}{p} \left(1 - \int dq \, \frac{dX_\varepsilon(q)}{dq} \left[q^{l+m} + q^{m+n} + q^{n+l} \right] \right.
$$

$$
+ \int dq \, \frac{dX_\varepsilon(q)}{dq} X_\varepsilon(q) \, q^{l+m+n}
$$

$$
\left. + \int dq_1 \, \frac{dX_\varepsilon(q_1)}{dq_1} \int_0^{q_2} dq_2 \, \frac{dX_\varepsilon(q_2)}{dq_2} \left[q_1^l q_2^{m+n} + q_1^m q_2^{n+l} + q_1^n q_2^{l+m} \right] \right)
$$

$$
\tag{39}
$$

The derivation is given in Appendix B. Since the observable O is local (in a suitably folded space), it is reasonable to expect that it equilibrates, and thus that the long-time limit of $\langle O(t) \rangle$ is equal to its equilibrium expectation value. Notice however that O is *not* the energy density, so that its thermalization is not guaranteed by general principles: this is the reason why the link between dynamic and static ultrametricity, although very plausible, is not completely proved by our derivation. Assuming the equilibration of O, we obtain

$$
P_\varepsilon^{(3)}(q_{12}, q_{13}, q_{23}) = \tfrac{1}{2} P_\varepsilon(q_{12}) \, x(q_{12}) \, \delta(q_{12} - q_{13}) \, \delta(q_{12} - q_{23})
$$

$$
+ \tfrac{1}{2} [P_\varepsilon(q_{12}) \, P_\varepsilon(q_{13}) \, \theta(q_{12} - q_{13}) \, \delta(q_{13} - q_{23})
$$

$$
+ \text{permutations}] \tag{40}
$$

This implies static ultrametricity with a weighting of the triangles as first found in ref. 29 from replica theory.

We shall now use again linear response to show that dynamical ultrametricity (and therefore, according to the above discussion, static ultrametricity) can be related to the property of "separability" defined in ref. 26.

In order to do so, let us introduce a different measure of overlap among states. Consider a system perturbed with the term $H_2(\mathbf{S}) = \sum_{x \in \mathscr{S}_1} J_x^{(2)} S_x S_{\mathscr{T}(x)}$. For this system we introduce a new static overlap probability function by the usual procedure, discussed in Section III. We

introduce two copies of the system, and define a new overlap Q^* between them by

$$Q^*(\mathbf{S}, \mathbf{S}') = \tfrac{1}{2} \sum_{x \in \mathscr{S}_1} J_x^{(2)}(S_x S'_{\mathscr{T}(x)} + S'_x S_{\mathscr{T}(x)}) \tag{41}$$

The new overlap probability function $P^*(q^*)$ is defined in complete analogy to (6) as the probability to find $Q^* = q^*$, at equilibrium. Correspondingly in dynamics one can define a new correlation function

$$C^*(t, t') = \tfrac{1}{2} \sum_{x \in \mathscr{S}_1} J_x^{(2)} \langle S_x(t) \, S_{\mathscr{T}(x)}(t') + S_x(t') \, S_{\mathscr{T}(x)}(t) \rangle \tag{42}$$

a new response function

$$R^*(t, t') = \frac{1}{2} \sum_{x \in \mathscr{S}_1} J_x^{(2)} \left[\frac{\delta \langle S_{\mathscr{T}(x)}(t) \rangle}{\delta h_x(t')} + \frac{\delta \langle S_x(t) \rangle}{\delta h_{\mathscr{T}(x)}(t')} \right] \tag{43}$$

and a new FDR $X^*(q^*)$ according to the definition (5), with C and R substituted by C^* and R^*. $C^*(t, t')$ represents the average correlation of a spin at time t with (a part of) its molecular field at time t'. As the correlation induced by H_2 is positive, $C^*(t, t')$ should be, as $C(t, t')$, a monotonically decreasing function of the time separation. We can then define for long times, a function

$$q^*(q) = \lim_{\substack{t, \, t' \to \infty \\ C(t, t') = q}} C^*(t, t')$$

which is monotonic, with the exception at most of constant or vertical parts. These parts would represent a completely different evolution of $C^*(t, t')$ and $C(t, t')$, what seems to us a quite strange possibility. This suggests that in statics pairs of states with equal q also have equal q^*. The algebraic expression of this property within the replica formalism has been discussed in ref. 26. If we admit this property of separability, we have $P^*(q^*) \, dq^* = P(q) \, dq$, and via stochastic stability $X^*(q^*(q)) = X(q)$. Integrating by parts over J_x as usual, we find the following expressions for C^* and R^*:

$$C^*(t, u) = \beta\varepsilon \left[\int_0^u ds \, C(t, s) \, R(u, s) + R(t, s) \, C(u, s) + \int_u^t ds \, R(t, s) \, C(s, u) \right]$$
$$R^*(t, u) = \beta\varepsilon \left[\int_u^t ds \, R(t, s) \, R(s, u) \right] \tag{44}$$

It is easy to check that in any correlation domain $\mathcal{D}_u$ where the correlation functions depend only on $h_u(t')/h_u(t)$, the function C^* depends on the same combinations of the time variables. If in addition we require that $R^*(t, t') = X(C)(\partial C^*(t, t')/\partial t')$, a detailed calculation shows that the only possibility is that $X(q)$ is a constant in $\mathcal{D}_u$.

We see then that if we assume the property of separability in statics, stochastic stability implies the existence of a unique dynamic function $X(q)$ independent of the particular correlation and response function used in the definition. This would corroborate the interpretation of $T/X(q)$ as a well-defined effective temperature for the system.[12] This conclusion depends on the validity, of separability in the statics. This is true in Long Range models within the framework of the mean-field theory, although even in these models a rigorous proof is still lacking. Unfortunately, we have not been able to find a way to prove this property in short range systems, although we feel that it should be possible to advance some arguments in favor of it, similar to the ones used above. The connection between separability and stochastic stability is an interesting point which needs to be clarified by further work.

VII. SUMMARY AND COMMENTS

We have thus developed the linear response theory for glassy systems, based on the equilibration of the free energy density. The introduction of local random perturbations of the Hamiltonian, treated in linear response, naturally leads to a new equilibrium order parameter function $\tilde{P}(q)$ and to a new fluctuation dissipation ratio $\tilde{X}(q)$, which are obtained as the limits of the corresponding quantities when the perturbation gets small. From a theoretical point of view these two quantities possess appealing properties:

• They are directly related by $d\tilde{X}/dq = \tilde{P}(q)$. Thus the link between the static equilibrium order parameter and the out of equilibrium dynamics is clear, and $\tilde{P}(q)$ can be measured by performing an off equilibrium dynamical measurement.

• If the original Hamiltonian H_0 has some quenched-in disorder, the static order parameter function $\tilde{P}(q)$ is self-averaging with respect to this disorder.

• The order parameter function $\tilde{P}(q)$ is defined by the values of extensive, thermodynamic quantities.

Therefore linear response allows to define a single, self averaging, order parameter function which is able to describe both the equilibrium properties and the off-equilibrium dynamics.

If stochastic stability holds, i.e., if the average correlation functions are continuous with respect to small perturbations of the system, $\tilde{P}(q)$ is equal to the overlap probability function $P(q)$ of the unperturbed system, up to some simple effects of global symmetries which we have discussed. We have thus found the root of the formal identities between static and off-equilibrium dynamical quantities, often empirically found in the analysis of specific models. In general, if the system has a symmetry which is lifted by the perturbation, like a global Ising symmetry for a spin glass in zero field, then the new order parameter $\tilde{P}(q)$ is equal to the usual overlap probability distribution in the presence of an infinitesimal breaking. Our discussion implies that (and explains why) the function $X(q)$ verifies the properties of a cumulative probability distribution.

The discussion has been extended to more complex characterization of the statistics of the equilibrium states. Stochastic stability allows to infer properties of the dynamics from properties of the statics and vice-versa. In this way we have argued, modulo some technical assumptions, that if *static* separability holds, i.e., if one can not define different notions of overlap yielding different predictions on the relation among the states, then *dynamically* the fluctuation dissipation ratio is a uniquely defined function independent on the particular correlation and response functions used in the definition. This suggests that ultrametricity should hold in all stochastically stable systems, both in the out of equilibrium dynamics and in the equilibrium properties.

To conclude, let us stress the important experimental implications of this work. Whenever the values of the energy density can be considered close to their equilibrium, the order parameter function $P(q)$ can be obtained from the determination of the function $X(q)$ in an aging experiment. This requires separate measurements of the response and the fluctuation, which are in principle both accessible to experimental determination. Although generically valid for infinite time in short range systems, depending on the specific situation, the energy density equilibration can be fulfilled or fail on laboratory time scales. In structural glasses for example, the relaxation time for energy density is so large that one is never probing the time region to which our analysis apply. In spin glasses conversely it is generally believed that energy densities are asymptotically close to equilibrium. An experimental determination of $X(q)$ in spin glasses would, be therefore of the utmost interest, being equivalent to the determination of $P(q)$.

APPENDIX A

We now derive the equalities (11) and (12). Using the fact that the $J_{i_1,\ldots,i_p}$ are Gaussian independent variables, we get from an integration by parts the following form of the canonical average of H_p^{LR}:

$$
\begin{aligned}
E_J \langle H_p^{\mathrm{LR}} \rangle &= E_J \left[\sum_{i_1 < \cdots < i_p} J_{i_1, \dots, i_p} \langle S_{i_1} \cdots S_{i_p} \rangle \right] \\
&= \frac{p!}{N^{p-1}} \sum_{i_1 < \cdots < i_p} E_J \left[\frac{\partial}{\partial J_{i_1, \dots, i_p}} \langle S_{i_1} \cdots S_{i_p} \rangle \right] \\
&= -\frac{\beta \varepsilon p!}{N^{p-1}} \sum_{i_1 < \cdots < i_p} E_J [1 - \langle S_{i_1} \cdots S_{i_p} \rangle^2] \\
&= -N \beta \varepsilon \left(1 - \int dp\, P(q)\, q^p \right)
\end{aligned}
\tag{A1}
$$

We obtain analogously in Langevin dynamics:

$$
E_J \langle H_p^{\mathrm{LR}}(t) \rangle = \frac{p!}{N^{p-1}} \sum_{i_1 < \cdots < i_p} E_J \left[\frac{\partial}{\partial J_{i_1, \dots, i_p}} \langle S_{i_1}(t) \cdots S_{i_p}(t) \rangle \right]
\tag{A2}
$$

By considering the expression (3) of the Martin–Siggia–Rose action we obtain

$$
\begin{aligned}
E_J \langle H_p^{\mathrm{LR}}(t) \rangle &= \frac{\varepsilon p!}{N^{p-1}} \int_0^t dt' \sum_{i_1 < \cdots < i_p} p \langle S_{i_1}(t) \cdots S_{i_p}(t)\, i\hat{S}_{i_1}(t')\, S_{i_2}(t') \cdots S_{i_p}(t') \rangle \\
&= N\varepsilon \int_0^t dt'\, p C^{p-1}(t, t')\, R(t, t')
\end{aligned}
\tag{A3}
$$

As stated in the text, equation (A3) holds in a much more general dynamical context. Consider a general relaxational dynamics in which spins are updated according to the local field, e.g.,

$$
S_x(t + \delta t) = f(h_x(t), \eta_x(t))
\tag{A4}
$$

where $h_x(t)$ is the molecular field at time t (which depends on the other spins and on the quenched variables) and $\eta_x(t)$ represents the thermal noise. $S_x(t)$ depends on $J_{i_1, \dots, i_p}$ only through its implicit dependence on $h_x(t')$ at all previous times. Therefore,

$$
\begin{aligned}
E_J \langle J_{i_1, \dots, i_p} & S_{i_1}(t) \cdots S_{i_p}(t) \rangle \\
&= E_J(J^2)\, E_J \frac{\partial}{\partial J_{i_1, \dots, i_p}} \langle S_{i_1}(t) \cdots S_{i_p}(t) \rangle \\
&= E_J(J^2)\, E_J \int_0^t dt' \sum_j \left\langle \frac{\partial h_j(t')}{\partial J_{i_1, \dots, i_p}} \frac{\partial}{\partial h_j(t')} S_{i_1}(t) \cdots S_{i_p}(t) \right\rangle
\end{aligned}
\tag{A5}
$$

Now it is easy to see that

$$\frac{\partial h_j(t')}{\partial J_{i_1,\dots,i_p}} = \sum_{\ell=1}^{p} \delta_{j,i_\ell} S_{i_1}(t') \cdots \{S_{i_\ell}(t')\} \cdots S_{i_p}(t') \tag{A6}$$

where $S_1 \cdots \{S_\ell\} \cdots S_p$ means that S_ℓ does *not* appear in the product. We thus find the relation (A3).

APPENDIX B

In this appendix we discuss the technical derivation of formulae (37) and (38), together with the the prove of (39) if dynamical ultrametricity holds. Although considerably more involved the derivations parallel the ones for the $P(q)$ and $X(q)$ without adding new physical insight. In order to simplify the exposition we introduce a multi-index notation

$$\begin{aligned}
i(x) &= \{x, \mathcal{T}_1^{(p)}(x),\dots, \mathcal{T}_{l-1}^{(p)}(x)\} \\
j(x) &= \{\mathcal{T}_l^{(p)}(x),\dots, \mathcal{T}_{l+m-1}^{(p)}(x)\} \\
k(x) &= \{\mathcal{T}_{l+m}^{(p)}(x),\dots, \mathcal{T}_{p-1}^{(p)}(x)\}
\end{aligned} \tag{B1}$$

and

$$\begin{aligned}
\mathbf{S}_{i(x)} &= S_x S_{\mathcal{T}_1^{(p)}(x)}\dots, S_{\mathcal{T}_{l-1}^{(p)}(x)} \\
\mathbf{S}_{j(x)} &= S_{\mathcal{T}_l^{(p)}(x)}\dots, S_{\mathcal{T}_{m+l-1}^{(p)}(x)} \\
\mathbf{S}_{k(x)} &= S_{\mathcal{T}_{m+l}^{(p)}(x)}\dots, S_{\mathcal{T}_{p-1}^{(p)}(x)}
\end{aligned} \tag{B2}$$

In these notations one has

$$H_p^{(1)} = \sum_x J_x \mathbf{S}_{i(x)} \mathbf{S}_{j(x)} \tag{B3}$$

$$H_p^{(2)} = \sum_x J_x \mathbf{S}_{j(x)} \mathbf{S}_{k(x)} \tag{B4}$$

$$O = \sum_x J_x J'_x \mathbf{S}_{i(x)} \mathbf{S}_{k(x)} \tag{B5}$$

If we compute the average of O in statics we obtain

$$
\begin{aligned}
E_J E_{J'} \langle O \rangle &= \sum_{x \in \mathscr{S}_1} E_J E_{J'} \frac{\partial}{\partial J_x} \frac{\partial}{\partial J'_x} \langle \mathbf{S}_{i(x)} \mathbf{S}_{k(x)} \rangle \\
&= \beta^2 \varepsilon \varepsilon' E_J E_{J'} (1 - \langle \mathbf{S}_{i(x)} \mathbf{S}_{k(x)} \rangle^2 - \langle \mathbf{S}_{j(x)} \mathbf{S}_{k(x)} \rangle^2 - \langle \mathbf{S}_{i(x)} \mathbf{S}_{j(x)} \rangle^2 \\
&\quad + 2 \langle \mathbf{S}_{i(x)} \mathbf{S}_{k(x)} \rangle \langle \mathbf{S}_{j(x)} \mathbf{S}_{k(x)} \rangle \langle \mathbf{S}_{i(x)} \mathbf{S}_{j(x)} \rangle)
\end{aligned}
\tag{B6}
$$

which, inserting the definition of $P(q)$ and $P^{(3)}(q, q', q'')$, reduces to (37).

In dynamics as well the derivation can be simplified by the use of a smart notation. The perturbations H_1 and H_2 induce the following terms respectively in the dynamical action:

$$
\int_0^t du \sum_{x \in \mathscr{S}_1} J_x \sum_{r=0}^{l+m-1} [S_x(u) \cdots \{S_{\mathscr{T}_r^{(p)}(x)}(u)\} \cdots S_{\mathscr{T}_{l+m-1}^{(p)}(x)}(u)] \, i\hat{S}_{\mathscr{T}_r^{(p)}(x)}(u)
\tag{B7}
$$

and

$$
\int_0^t du \sum_{x \in \mathscr{S}_1} J'_x \sum_{r=l}^{p-1} [S_{\mathscr{T}_l^{(p)}(x)}(u) \cdots \{S_{\mathscr{T}_r^{(p)}(u)}\} \cdots S_{\mathscr{T}_{p-1}^{(p)}}(u)] \, i\hat{S}_{\mathscr{T}_r^{(p)}(u)}
\tag{B8}
$$

Let us now define

$$
\begin{aligned}
i\hat{\mathbf{S}}_{i(x)} &= \sum_{r=0}^{l-1} [S_x(u) \cdots \{S_{\mathscr{T}_r^{(p)}(u)}\} \cdots S_{\mathscr{T}_{l-1}^{(p)}}(u)] \, i\hat{S}_{\mathscr{T}_r^{(p)}(u)} \\
i\hat{\mathbf{S}}_{j(x)} &= \sum_{r=l}^{l+m-1} [S_{\mathscr{T}_l^{(p)}(x)}(u) \cdots \{S_{\mathscr{T}_r^{(p)}(x)}(u)\} \cdots S_{\mathscr{T}_{l+m-1}^{(p)}(x)}(u)] \, i\hat{S}_{\mathscr{T}_r^{(p)}(x)}(u) \\
i\hat{\mathbf{S}}_{k(x)} &= \sum_{r=l+m}^{p-1} [S_{\mathscr{T}_{l+m}^{(p)}(x)}(u) \cdots \{S_{\mathscr{T}_r^{(p)}(x)}(u)\} \cdots S_{\mathscr{T}_{p-1}^{(p)}(x)}(u)] \, i\hat{S}_{\mathscr{T}_r^{(p)}(x)}(u)
\end{aligned}
\tag{B9}
$$

With this notation the terms (B7), and (B8) are respectively written as

$$
\int_0^t du \sum_{x \in \mathscr{S}_1} J_x [i\hat{\mathbf{S}}_{i(x)}(u) \, \mathbf{S}_{j(x)}(u) + i\hat{\mathbf{S}}_{j(x)}(u) \, \mathbf{S}_{i(x)}(u)]
$$

$$
\int_0^t du \sum_{x \in \mathscr{S}_1} J'_x [i\hat{\mathbf{S}}_{j(x)}(u) \, \mathbf{S}_{k(x)}(u) + i\hat{\mathbf{S}}_{k(x)}(u) \, \mathbf{S}_{j(x)}(u)]
\tag{B10}
$$

Evaluating $\langle O(t) \rangle$ by integration by parts we obtain

$$
\begin{aligned}
\langle O(t) \rangle = \varepsilon \varepsilon' \sum_{x \in \mathscr{S}_1} \int_0^t du \int_0^t dv \langle \mathbf{S}_{i(x)}(t) \, \mathbf{S}_{k(x)}(t) \\
\times [\, i\hat{\mathbf{S}}_{i(x)}(u) \, \mathbf{S}_{j(x)}(u) + i\hat{\mathbf{S}}_{j(x)}(u) \, \mathbf{S}_{i(x)}(u) \,] \\
\times [\, i\hat{\mathbf{S}}_{j(x)}(v) \, \mathbf{S}_{k(x)}(v) + i\hat{\mathbf{S}}_{k(x)}(v) \, \mathbf{S}_{j(x)}(v) \,] \rangle
\end{aligned}
\tag{B11}
$$

Using the clustering property for far-away sites for small ε, ε', we get the relation

$$
\langle \mathbf{S}_{i(x)}(t) \, i\hat{\mathbf{S}}_{i(x)}(u) \rangle = l C^{l-1}(t, u) \, R(t, u)
\tag{B12}
$$

and analogous ones for the averages involving the multi-indices $j(x)$ and $k(x)$, which collected together give Eq. (38).

Let us now sketch the derivation of Eq. (39) from dynamical ultrametricity. Let us simplify the notations in Eq. (38) by writing, for $r = l, m, n$,

$$
\begin{aligned}
C_r(t, u) &= C^r(t, u) \\
R_r(t, u) &= r C^{r-1}(t, u) \, R(t, u)
\end{aligned}
\tag{B13}
$$

In this way Eq. (38) reads:

$$
\begin{aligned}
\frac{E_J E_{J'} \langle O(t) \rangle}{\beta^2 \varepsilon \varepsilon' N} = \frac{1}{p} \int_0^t dv \int_0^t du [\, R_l(t, v) \, C_m(v, u) \, R_n(t, u) \\
+ C_l(t, v) \, R_m(v, u) \, R_n(t, u) + C_l(t, v) \, R_m(u, v) \, R_n(t, u) \,]
\end{aligned}
\tag{B14}
$$

For long times we can write $R_r(t, u) \approx X_\varepsilon(C(t, u)) \, \partial_u C_r(t, u)$, and the l.h.s. of Eq. (B14) becomes:

$$
\begin{aligned}
\frac{1}{p} \int_0^t du \int_0^u dv [\, X_\varepsilon(C(t, v)) \, \partial_v C_l(t, v) \, C_m(v, u) \, X_\varepsilon(C(t, u)) \, \partial_u C_n(t, u) \\
+ C_l(t, v) \, X_\varepsilon(C(u, v)) \, \partial_u C_m(u, v) \, X_\varepsilon(C(t, u)) \, \partial_u R_n(t, u) \,] + [\, n \leftrightarrow l \,]
\end{aligned}
\tag{B15}
$$

The fact that $X_\varepsilon(C)$ is constant within each crossover domain allows as to manipulate this equation as if the ultrametric relation among the

correlation function were to hold for all time involved. This can be realized as the constancy of X_ε implies that in each domain one has to integrate a total derivative and the integrals can be performed explicitly.

We now introduce the shorthand notations

$$(\partial C(t, u)/\partial u)\, du \mapsto dC(t, u) \tag{B16}$$

$$nC^{n-1}(t, u)\, dC(t, u) \mapsto dC_n(t, u) \tag{B17}$$

and similarly for the other differential. Using this notation and the constancy of X_ε, we find that the first term of Eq. (B15) can be written as

$$\int_0^1 dC_n(t, u)\, X_\varepsilon(C(t, u)) \int_0^{C(t, u)} dC_l(t, v)\, C_m(t, v)\, X_\varepsilon(C(t, v)) \tag{B18}$$

while the second one is given by

$$\int_0^1 dC_m(t, v)\, X_\varepsilon(C(t, v)) \int_0^{C(t, v)} dC_n(t, u)\, C_l(u, v)\, X_\varepsilon(C(u, v)) + [m \leftrightarrow n] \tag{B19}$$

The extrema of integration refer to the variable C. Integrating by parts and collecting all terms we find Eq. (38).

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